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The Spin Interaction of a Dirac Particle in an Aharonov-Bohm Potential in First Order Scattering

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Abstract For a Dirac particle in an Aharonov-Bohm (AB) potential, it is shown that the spin interaction (SI) operator which governs the transitions in the spin sector of the first order S-matrix is related to one of the generators of rotation in the spin space of the particle. This operator, which is given by the projection of the spin operator $\boldsymbol{\Sigma}$ along the direction of the total momentum of the system, and the two operators constructed from the projections of the $\boldsymbol{\Sigma}$ operator along the momentum transfer and the z-directions close the $SU(2)$ algebra. It is suggested, then, that these two directions of the total momentum and the momentum transfer form some sort of natural intrinsic directions in terms of which the spin dynamics of the scattering process at first order can be formulated conveniently. A formulation and an interpretation of the conservation of helicity at first order using the spin projection operators along these directions is presented.

Keywords Aharonov-Bohm

1 Introduction

The perturbative treatments of the Aharonov-Bohm (AB) effect [1] received much interest in the literature as a result of the observation [5, 7] that perturbation theory fails when applied to this theory: The first order scattering amplitude was found to differ from the exact one when the latter is expanded to this order, and the second order was shown to diverge. Many approaches were suggested in the literature to remedy this problem for non-relativistic particles, examples of which are [2, 4, 6, 9–14]. When the spin degree of freedom is taken into account [8], it was shown that perturbation theory works for both non-relativistic [13] and relativistic particles [3, 16, 17]; thanks to the spin-magnetic moment interaction.

In a recent work [15], some algebraic properties of the interaction Hamiltonian in first order perturbation theory of a Dirac particles in an AB potential were reported and the general structure of the transition matrix at this order based on these algebraic properties was

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discussed. In this work, the spin interaction (SI) (the interaction that remains after integrating out the space degrees of freedom) in the first order S-matrix of a Dirac particle in the AB potential is investigated. It is shown that this SI is related to the generators of rotation in the spin space of the particle. It is suggested that the spin dynamics at this order can be formulated in terms of the projections of the spin operators along two directions characteristic of the scattering process; the total momentum and the momentum transfer directions. Conservation of helicity at this order is formulated using these operators.

2 Algebra of the Spin Interaction

A Dirac particle in an electromagnetic field is governed by the Hamiltonian ($\hbar = c = 1$):

$$H = H_0 + H_{\text{int}}, \tag{1}$$

where

$$H_0 = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m, \tag{2}$$

and

$$H_{\text{int}} = eA_0 - e\boldsymbol{\alpha} \cdot \mathbf{A}. \tag{3}$$

Here, e is the charge of the particle, $\alpha_i = \beta\gamma_i$ and $\beta = \gamma_4$. The γ's are the Dirac matrices: $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. The first order S-matrix element for the particle is:

$$S_{fi}^{(1)} = -i \int d^4x\, \bar{\psi}_f(x)(e\gamma_\mu A^\mu)\psi_i(x). \tag{4}$$

For the AB-potential Aharonov and Bohm [1], we have:

$$A_0 = 0, \tag{5}$$

and

$$\mathbf{A} = \frac{\Phi}{2\pi\rho}\hat{\epsilon}_\varphi, \tag{6}$$

where $\rho = \sqrt{x^2 + y^2}$, $\hat{\epsilon}_\varphi$ is the unit vector in the φ-direction, and Φ is the flux through the AB tube. The free Dirac wave functions are taken as:

$$\psi^s(x) = \frac{1}{L^{3/2}}\left(\frac{m}{E}\right)^{1/2} u(p,s)e^{-ip.x} \tag{7}$$

with

$$u(p,s) = \left(\frac{E+m}{2m}\right)^{1/2} \omega(p,s). \tag{8}$$

Plugging the above vector potential into the matrix element, (4), carrying out the time and space integrals (taking $\mathbf{p_i} = (p, 0, 0)$, $\mathbf{p_f} = (p\cos\theta, p\sin\theta, 0)$, θ being the scattering angle) we get the first order S-matrix element as:

$$S_{fi}^{(1)} = -4\pi^2\Delta|N|^2\delta(E_f - E_i)u_f^\dagger(p_f, s_f)\left(\frac{\alpha_1 q_2 - \alpha_2 q_1}{q^2}\right)u_i(p_i, s_i). \tag{9}$$

Here, $\Delta = -e\Phi/2\pi$ (in perturbative calculations $0 < \Delta < 1$). $|N|^2$ is the normalization constant appearing in (7) (times a factor of L coming from the z-integral), and $\mathbf{q} = \mathbf{p_f} - \mathbf{p_i}$ is the momentum transfer. Since we will be focusing on the spin interaction in this work, it is convenient to write the above matrix element as

$$S_{fi}^{(1)} = -4\pi^2 \Delta |N|^2 \delta(E_f - E_i) \frac{M^{(1)}}{q} \tag{10}$$

with $M^{(1)}$ defined—using Dirac notation—as

$$M^{(1)} = \left\langle \mathbf{p_f}; s_f \left| \frac{\alpha_1 q_2 - \alpha_2 q_1}{q} \right| \mathbf{p_i}; s_i \right\rangle. \tag{11}$$

Noting that $\alpha_i = \gamma_5 \Sigma_i$, where $\Sigma_i = \frac{i}{2}[\gamma_i, \gamma_j]$, $(i, j = 1..3)$, and $i\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, we define the Hermitian spin interaction operator (SI)

$$K \equiv \left(\frac{\alpha_1 q_2 - \alpha_2 q_1}{q} \right) = \gamma_5 \boldsymbol{\Sigma}.\hat{\mathbf{k}} \tag{12}$$

with $\hat{\mathbf{k}}$ ($\hat{\mathbf{k}} = \frac{\mathbf{p_f}+\mathbf{p_i}}{|\mathbf{p_f}+\mathbf{p_i}|} = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, 0)$), being a unit vector bisecting the scattering angle θ and lying along the sum of the initial and final momenta of the particle (see Fig. 1 below). Therefore, $M^{(1)}$ reads now

$$M^{(1)} = \langle \mathbf{p_f}; s_f | \gamma_5 \boldsymbol{\Sigma}.\hat{\mathbf{k}} | \mathbf{p_i}; s_i \rangle. \tag{13}$$

The following algebra of the SI operator K (more precisely , of $\gamma_5 K$) can be easily verified ($\hat{\mathbf{q}} = \frac{\mathbf{q}}{q}$):

$$\begin{aligned} [\boldsymbol{\Sigma}.\hat{\mathbf{k}}, \boldsymbol{\Sigma}.\hat{\mathbf{q}}] &= 2i\Sigma_3, \\ [\Sigma_3, \boldsymbol{\Sigma}.\hat{\mathbf{k}}] &= 2i\boldsymbol{\Sigma}.\hat{\mathbf{q}}, \\ [\boldsymbol{\Sigma}.\hat{\mathbf{q}}, \Sigma_3] &= 2i\boldsymbol{\Sigma}.\hat{\mathbf{k}}, \end{aligned} \tag{14}$$

and

$$\{\boldsymbol{\Sigma}.\hat{\mathbf{k}}, \boldsymbol{\Sigma}.\hat{\mathbf{q}}\} = \{\Sigma_3, \boldsymbol{\Sigma}.\hat{\mathbf{k}}\} = \{\boldsymbol{\Sigma}.\hat{\mathbf{q}}, \Sigma_3\} = 0. \tag{15}$$

Moreover, one can easily verify that

$$(\boldsymbol{\Sigma}.\hat{\mathbf{k}})^2 = (\boldsymbol{\Sigma}.\hat{\mathbf{q}})^2 = (\Sigma_3)^2 = I. \tag{16}$$

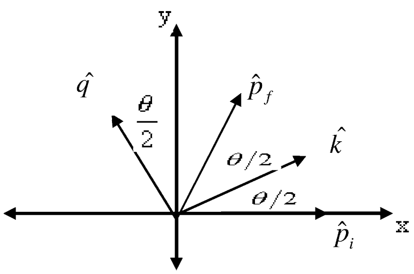

Fig. 1 Scattering diagram in the xy-plane

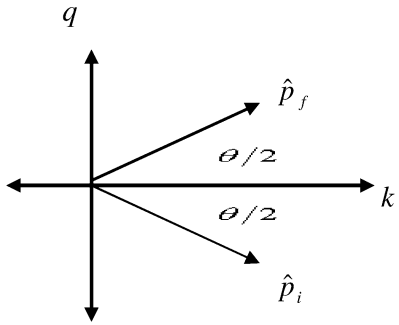

Fig. 2 Scattering diagram in the k–q plane

The above algebra, which is just the $SU(2)$ algebra says that the two operators $\mathbf{\Sigma}.\hat{\mathbf{k}}$ and $\mathbf{\Sigma}.\hat{\mathbf{q}}$ along with Σ_3 are the generators of rotation in the spin space of the particle. To get a further insight into this, note first that the two unit vectors $\hat{\mathbf{q}}$ and $\hat{\mathbf{k}}$—which can be defined for any similar scattering process—are functions of the scattering angle θ and are mutually orthogonal. They are the analogues of the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ polar unit vectors in the position space. Recalling that $\hat{\mathbf{q}}$ and $\hat{\mathbf{k}}$ lie respectively, along the momentum transfer and the total momentum directions, then it is as if the dynamics of any scattering process define some sort of intrinsic natural coordinates for the process at first order. Therefore, the set of operators $\mathbf{\Sigma}.\hat{\mathbf{k}}$, $\mathbf{\Sigma}.\hat{\mathbf{q}}$ and Σ_3 close the $SU(2)$ algebra in any similar scattering process, and are merely the generators of rotation expressed in terms of the "intrinsic" coordinates. What is special with the AB potential, on the other hand, is the fact that the SI operator of this potential is related (note that $[\gamma_5, \Sigma_i] = 0$, $\forall i = 1\ldots3$) to one of these generators, namely, $\mathbf{\Sigma}.\hat{\mathbf{k}}$. This fact has some consequences regarding the spin dynamics in the transition, in particular, the conservation of helicity, as we will show below. To start with, recall that the AB potential conserves helicity [17], now express the conserved helicity operators and states in terms of the operators $\mathbf{\Sigma}.\hat{\mathbf{k}}$ and $\mathbf{\Sigma}.\hat{\mathbf{q}}$ and their eigenstates. Indeed, the helicity operators $\mathbf{\Sigma}.\hat{\mathbf{p}}_\mathbf{i}$ and $\mathbf{\Sigma}.\hat{\mathbf{p}}_\mathbf{f}$ of the incident and outgoing particles, respectively, can be expanded as:

$$\begin{aligned} \mathbf{\Sigma}.\hat{\mathbf{p}}_\mathbf{i} &= \cos\tfrac{\theta}{2}\mathbf{\Sigma}.\hat{\mathbf{k}} - \sin\tfrac{\theta}{2}\mathbf{\Sigma}.\hat{\mathbf{q}}, \\ \mathbf{\Sigma}.\hat{\mathbf{p}}_\mathbf{f} &= \cos\tfrac{\theta}{2}\mathbf{\Sigma}.\hat{\mathbf{k}} + \sin\tfrac{\theta}{2}\mathbf{\Sigma}.\hat{\mathbf{q}}. \end{aligned} \tag{17}$$

Note the symmetry involved in these two expressions, which is just a reflection of the symmetry of Fig. 2. With the above expressions at hand, it is easy now to verify with the aid of the algebra given by (14–16) the following relations:

$$\begin{aligned} K\mathbf{\Sigma}.\hat{\mathbf{p}}_\mathbf{i}K &= \mathbf{\Sigma}.\hat{\mathbf{p}}_\mathbf{f}, \\ K\mathbf{\Sigma}.\hat{\mathbf{p}}_\mathbf{f}K &= \mathbf{\Sigma}.\hat{\mathbf{p}}_\mathbf{i}. \end{aligned} \tag{18}$$

Recalling that the operator K is unitary in addition to being Hermitian, the above relations say that the spin interaction operator is a unitary transformation that relates the incident and outgoing particle's helicity operators so that they form a couple of unitary equivalent observables. The immediate consequence of this is the conservation of helicity. A detailed investigation of this issue will be given in the next section.

3 Conservation of Helicity at First Order

In any helicity-conserving theory, the conservation of the helicity in first order scattering is manifested by the vanishing of the matrix element:

$$\langle \mp; \hat{\mathbf{p}}_\mathbf{f}|K|\hat{\mathbf{p}}_\mathbf{i}; \pm\rangle = 0, \tag{19}$$

where $|\hat{\mathbf{p}}_\mathbf{i}; \pm\rangle$ ($|\hat{\mathbf{p}}_\mathbf{f}; \pm\rangle$) are the eigenstates of $\boldsymbol{\Sigma}.\hat{\mathbf{p}}_\mathbf{i}$ ($\boldsymbol{\Sigma}.\hat{\mathbf{p}}_\mathbf{f}$) with eigenvalues ± 1. We will show now that (18) will naturally lead to the result (19). First, put (18) into the form

$$\boldsymbol{\Sigma}.\hat{\mathbf{p}}_\mathbf{f} K \boldsymbol{\Sigma}.\hat{\mathbf{p}}_\mathbf{i} = K \tag{20}$$

then have both sides acting on the incident particle spinor to get:

$$\boldsymbol{\Sigma}.\hat{\mathbf{p}}_\mathbf{f} K|\hat{\mathbf{p}}_\mathbf{i}; \pm\rangle = \pm K|\hat{\mathbf{p}}_\mathbf{i}; \pm\rangle. \tag{21}$$

The state $K|\hat{\mathbf{p}}_\mathbf{i}; \pm\rangle$ is an eigenstate of $\boldsymbol{\Sigma}.\hat{\mathbf{p}}_\mathbf{f}$ with eigenvalues ± 1, i.e. SI K does not flip the helicity of the initial state. This, in turn, guarantees that (19) holds: Using (20) one has:

$$\begin{aligned}\langle \mp; \hat{\mathbf{p}}_\mathbf{f}|K|\hat{\mathbf{p}}_\mathbf{i}; \pm\rangle &= \langle \mp, \hat{\mathbf{p}}_\mathbf{f}|\boldsymbol{\Sigma}.\mathbf{p}_\mathbf{f} K \boldsymbol{\Sigma}.\mathbf{p}_\mathbf{i}|\hat{\mathbf{p}}_\mathbf{i}; \pm\rangle \\ &= -\langle \mp; \hat{\mathbf{p}}_\mathbf{f}|K|\hat{\mathbf{p}}_\mathbf{i}; \pm\rangle,\end{aligned} \tag{22}$$

where we have allowed $\boldsymbol{\Sigma}.\hat{\mathbf{p}}_\mathbf{i}$ and $\boldsymbol{\Sigma}.\hat{\mathbf{p}}_\mathbf{f}$ to act on their eigenstates. Therefore,

$$\langle \mp; \hat{\mathbf{p}}_\mathbf{f}|K|\hat{\mathbf{p}}_\mathbf{i}; \pm\rangle = 0. \tag{23}$$

The present formulation encourages one to formulate conservation of helicity in the $\hat{\mathbf{k}}$–$\hat{\mathbf{q}}$ plane (see Fig. 2). The symmetry in this figure suggests that conservation of helicity implies the invariance of the components of both $\boldsymbol{\Sigma}.\hat{\mathbf{p}}_\mathbf{i}$ and $\boldsymbol{\Sigma}.\hat{\mathbf{p}}_\mathbf{f}$ along the $\hat{\mathbf{k}}$-axis, and the flipping of their components along the $\hat{\mathbf{q}}$-axis. Indeed, the algebra given by (14–16) has the consequence

$$K\boldsymbol{\Sigma}.\hat{\mathbf{k}}K = \boldsymbol{\Sigma}.\hat{\mathbf{k}}, \tag{24}$$

$$K\boldsymbol{\Sigma}.\hat{\mathbf{q}}K = -\boldsymbol{\Sigma}.\hat{\mathbf{q}}. \tag{25}$$

The above result can also be read off from the expansion of the eigenstates of $\boldsymbol{\Sigma}.\hat{\mathbf{p}}_\mathbf{i}$ and $\boldsymbol{\Sigma}.\hat{\mathbf{p}}_\mathbf{f}$ in terms of the eigenstates of $\boldsymbol{\Sigma}.\hat{\mathbf{k}}$:

$$\begin{aligned}|\hat{\mathbf{p}}_\mathbf{i}; \pm\rangle &= \left(\cos\frac{\theta}{4} \mp \sin\frac{\theta}{4}\boldsymbol{\Sigma}.\hat{\mathbf{q}}\right)|\hat{\mathbf{k}}; \pm\rangle, \\ |\hat{\mathbf{p}}_\mathbf{f}; \pm\rangle &= \left(\cos\frac{\theta}{4} \pm \sin\frac{\theta}{4}\boldsymbol{\Sigma}.\hat{\mathbf{q}}\right)|\hat{\mathbf{k}}; \pm\rangle.\end{aligned} \tag{26}$$

The invariance of the $\hat{\mathbf{k}}$-component is established immediately:

$$\langle \hat{\mathbf{k}}, \pm|\hat{\mathbf{p}}_\mathbf{i}, \pm\rangle = \langle \hat{\mathbf{k}}, \pm|\hat{\mathbf{p}}_\mathbf{f}, \pm\rangle = \cos\frac{\theta}{4}, \tag{27}$$

where we have used the result

$$\langle \hat{\mathbf{k}}, \pm|\boldsymbol{\Sigma}.\hat{\mathbf{q}}|\hat{\mathbf{k}}, \pm\rangle = 0. \tag{28}$$

In a similar manner, if one expands using the $\boldsymbol{\Sigma}.\hat{\mathbf{q}}$ eigenstates, the flipping of the $\hat{\mathbf{q}}$-component can be deduced immediately.

4 Conclusions

The effective spin interaction SI in the first order S-matrix of a Dirac particle in an AB potential was expressed as $K = \gamma_5 \mathbf{\Sigma}.\hat{\mathbf{k}}$, where $\hat{\mathbf{k}} = \frac{\mathbf{p_f}+\mathbf{p_i}}{|\mathbf{p_f}+\mathbf{p_i}|} = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, 0)$ is a unit vector in the scattering plane that bisects the scattering angle θ. It was shown that the set of operators $\mathbf{\Sigma}.\hat{\mathbf{k}}$, $\mathbf{\Sigma}.\hat{\mathbf{q}}$ ($\hat{\mathbf{q}} = \frac{\mathbf{q}}{q}$, $\mathbf{q} = \mathbf{p_f} - \mathbf{p_i}$) and Σ_3 close the $SU(2)$ algebra. This means that $\mathbf{\Sigma}.\hat{\mathbf{k}}$ and $\mathbf{\Sigma}.\hat{\mathbf{q}}$ can be identified—in addition to Σ_3—as furnishing a representation of the generators of this group in the spin space of the particle. Noting that the vectors $\hat{\mathbf{k}}$ and $\hat{\mathbf{q}}$ lie along the total momentum and the momentum transfer directions, respectively, then, they form some sort of natural "intrinsic" coordinates for any similar scattering process at first order. Therefore, it is natural to have the projections of $\mathbf{\Sigma}$ along these directions close $SU(2)$ algebra. The fact that the spin interaction K in the AB potential is related to one of these generators was shown to make the transition in the AB case a unitary transformation with the helicity operators of the incident and outgoing particles forming a couple of unitary equivalent observables. This was shown, then, to guarantee conservation of helicity at first order as should be. Moreover, a new view of the conservation of helicity in the first order transition was suggested by expressing the helicity operators and eigenstates in terms of their components along the $\hat{\mathbf{k}}$ and $\hat{\mathbf{q}}$. It was shown, then, that conservation of helicity was manifested as the invariance of the component of $\mathbf{\Sigma}$ along the $\mathbf{k}$-axis and the flipping of its component along the $\mathbf{q}$-axis.

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